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Factorizing twists and the universal R -matrix of the Yangian $Y(\mathfrak{sl}_2)$

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Abstract. We give an explicit construction of the factorizing twists for the Yangian $Y(\mathfrak{sl}_2)$ in evaluation representations (not necessarily finite dimensional). The result is a universal expression for the factorizing twist that holds in all of these representations. The method is general enough to recover the universal R -matrix of $Y(\mathfrak{sl}_2)$ up to its character in the form specialized to generic evaluation representations. The method presented here is particularly amenable to generalizations because it involves only elementary operations applied to representations of the Yangian.

1. Introduction

Drinfel'd twists [1] have been applied very successfully to quantum integrable spin chains within the framework of the algebraic Bethe ansatz. It was shown by Maillet and Sanchez de Santos in [2] that the Yangian $Y(\mathfrak{sl}_2)$ and the quantized envelope of the affine Lie algebra $U_q(\widehat{\mathfrak{sl}}_2)$ admit factorizing twists in at least the fundamental evaluation representation. Such a factorizing twist F equips the quantum algebra with a new coproduct $\Delta_F = F \Delta F^{-1}$ which is cocommutative. In representations the effect is that tensor products of representations admit a change of basis such that all expressions originating from coproducts are symmetric under exchange of the tensor factors. The cocommutativity of the coproduct has provided a dramatic simplification of the algebraic Bethe ansatz. The application of factorizing twists has triggered an interesting development with far-reaching new results for correlation functions [3–5].

From a mathematical point of view these twists are particularly interesting because, if they exist, they factorize the universal R -matrix, $R = F_{21}^{-1} F_{12}$ (hence the name), and thus can be considered as more fundamental than the R -matrix itself. We demonstrate in this paper that, for evaluation representations, the twists can even be used to determine the universal R -matrix up to a factor. On the other hand, the factorizing twists provide the Yangian $Y(\mathfrak{sl}_2)$ with an additional, very restrictive structure which has not yet been fully exploited in the analysis of the algebra.

Recently, factorizing twists have been found for the evaluation representation of the Yangian $Y(\mathfrak{sl}_n)$ corresponding to the first fundamental representation of \mathfrak{sl}_n [6] and for all finite-dimensional evaluation representations of the Yangian $Y(\mathfrak{sl}_2)$ [7], though the diagonal part of the twist was not determined there. Here the term ‘diagonal part’ refers to the Gauss decomposition of the twist $F = F_0 F_-$ into a diagonal operator F_0 and a lower triangular F_- . Likewise the universal R -matrix can be Gauss decomposed $R = R_+ R_0 R_-$.

It has been known for some time [8] that the Yangian $Y(\mathfrak{sl}_2)$ is pseudo-triangular. In view of the results for finite-dimensional semi-simple Hopf algebras [9] it is natural to conjecture that the pseudo-triangularity of $Y(\mathfrak{sl}_2)$ might lead to the construction of factorizing twists at least in a certain class of representations. This hypothesis is supported by the results of [7], but there it was not shown explicitly how the twists factorize the (pseudo-)universal R -matrix.

The study of the factorizing twists in [7] used a modification of the functional Bethe ansatz. This method has the advantage that the calculation of the iterated twisted coproduct is manageable and that this coproduct is cocommutative by construction. Cocommutativity is actually a property of the polynomial interpolation which is employed in the functional Bethe ansatz. One of the disadvantages, however, is that the change of basis from the non-cocommutative situation to the cocommutative one, i.e. the factorizing twist in a particular representation, is given only implicitly. In [7] it was possible to identify the triangular part of the twist and to show that it agrees with the triangular part of the Gauss decomposition of the R -matrix of $Y(\mathfrak{sl}_2)$ which is ‘universal’ for all evaluation representations of $Y(\mathfrak{sl}_2)$ [10], but the diagonal part was not given explicitly.

In this paper, we present a conceptually very simple direct and completely explicit calculation of the factorizing twist for evaluation representations of $Y(\mathfrak{sl}_2)$ which are modelled using highest-weight representations of the corresponding Lie algebra \mathfrak{sl}_2 which are not necessarily finite dimensional. It allows us to show in detail how the twist factorizes the triangular and diagonal parts of the universal R -matrix on evaluation representations. This complements the result of [7] and confirms the conjecture that the factorizing twists come from a universal object. This is the first main result of this paper.

From our construction of the factorizing twist, it is also possible to recover the generic form of the R -matrix for evaluation representations up to a scalar factor, the so-called character [10], which depends on the given representations. Our method allows us to reproduce this ‘universal’ expression for the R -matrix which was derived in [10] where the quantum double $\mathcal{DY}(\mathfrak{sl}_2)$ of the Yangian was used to calculate it. Furthermore, the R -matrix obtained via the factorizing twist appears in a more natural form that is simplified compared with the expression derived via the quantum double.

Obviously, once the R -matrix is known for evaluation representations, the axioms of the quasi-triangular structure determine the R -matrix for tensor products of evaluation representations. Hence our method is sufficient to obtain R -matrices for all tensor products of evaluation representations as well. These cover, in particular, all finite-dimensional irreducible representations of $Y(\mathfrak{sl}_2)$ and those representations which are of interest in applications to integrable systems.

We wish to emphasize that the direct method presented here does not make use of any fusion methods, but rather determines the factorizing twist and the R -matrix in one step for all evaluation representations. The ideas employed here are simple enough to offer a chance for a generalization both to the trigonometric case where the algebra is $U_q(\widehat{\mathfrak{sl}}_2)$ and also to the cases of higher rank.

The paper is organized as follows. In section 2 we recall the basic properties of factorizing twists and fix our notation for the study of evaluation representations of the Yangian $Y(\mathfrak{sl}_2)$.

In section 3 we begin the derivation of the factorizing twist on a generic evaluation representation. We find that the triangular part of the twist is determined by the requirement that the coproduct of one of the algebra generators (here $\Delta D(u)$) in this representation be diagonal after twisting. This is based on the same idea as the modified functional Bethe ansatz in [7].

The diagonal part of the twist is discussed in section 4. We find that the diagonal part can be determined by the additional requirement that the twisted coproduct be cocommutative.

The result arises as a set of recursion relations for the coefficients of the diagonal part which can be solved in such a way that the resulting form of the diagonal part is independent of the particular representations.

In section 5 we demonstrate in detail how the twist factorizes the universal R -matrix of $Y(\mathfrak{sl}_2)$ on evaluation representations. We summarize how the different ways of twisting which are based on the diagonalization of different generators or on the use of different diagonal parts for the twist, are related. Finally, we determine the precise conditions for the existence of the factorizing twist on evaluation representations.

Section 6 contains a summary and comments on future directions of research and important open questions.

2. Preliminaries

2.1. Factorizing twists

In order to fix the notation let us first recall the key definitions and theorems concerning factorizing twists. They are due to Drinfel'd [1, 11] where more details can be found. We write $\mu, \eta, \Delta, \varepsilon, S$ for the product, unit, coproduct, co-unit and antipode, respectively, of a Hopf algebra.

Definition 2.1. A Hopf algebra \mathcal{A} is called quasi-triangular if there exists an invertible element $R \in \mathcal{A} \otimes \mathcal{A}$, called the universal R -matrix, which satisfies

$$(\Delta \otimes \text{id})(R) = R_{13}R_{23} \tag{2.1a}$$

$$(\text{id} \otimes \Delta)(R) = R_{13}R_{12} \tag{2.1b}$$

$$\Delta^{\text{op}}(a) = R \cdot \Delta(a) \cdot R^{-1} \tag{2.1c}$$

for all $a \in \mathcal{A}$. It is called triangular if, in addition,

$$R_{21} = R_{12}^{-1}. \tag{2.2}$$

Here R_{ij} denote, as usual, the actions of R on the different factors of $\mathcal{A} \otimes \mathcal{A} \otimes \mathcal{A}$.

Definition 2.2. Let \mathcal{A} be a Hopf algebra. An invertible element $F \in \mathcal{A} \otimes \mathcal{A}$ is called a co-unital 2-cocycle or Drinfel'd twist if it satisfies

$$(\varepsilon \otimes \text{id})(F) = 1 \tag{2.3a}$$

$$(\text{id} \otimes \varepsilon)(F) = 1 \tag{2.3b}$$

$$F_{12} \cdot (\Delta \otimes \text{id})(F) = F_{23} \cdot (\text{id} \otimes \Delta)(F). \tag{2.3c}$$

Theorem 2.3 (Drinfel'd). Let \mathcal{A} be a quasi-triangular Hopf algebra and $F \in \mathcal{A} \otimes \mathcal{A}$ be a co-unital 2-cocycle. Then the algebra of \mathcal{A} together with the operations

$$\Delta_F(a) := F \cdot \Delta(a) \cdot F^{-1} \tag{2.4a}$$

$$S_F(a) := u \cdot S(a) \cdot u^{-1} \quad u := \mu(\text{id} \otimes S)(F) \tag{2.4b}$$

$$R_F := F_{21} \cdot R \cdot F^{-1} \tag{2.4c}$$

and the old co-unit ε , forms a quasi-triangular Hopf algebra \mathcal{A}_F .

The cocycle condition (2.3c) is required in order to make the twisted coproduct Δ_F co-associative so that one obtains a Hopf algebra rather than just a quasi-Hopf algebra.

Definition 2.4. Let \mathcal{A} be a quasi-triangular Hopf algebra with a co-unital 2-cocycle $F \in \mathcal{A} \otimes \mathcal{A}$. F is called a factorizing twist if $R_F = 1 \otimes 1$ in theorem 2.3, i.e.

$$R_{12} = F_{21}^{-1} \cdot F_{12}. \tag{2.5}$$

Remark 2.5.

- (a) If a quasi-triangular Hopf algebra \mathcal{A} admits a factorizing twist, the twisted coproduct Δ_F is cocommutative.
- (b) In this case the Hopf algebra is triangular since its universal R -matrix satisfies

$$R_{21} = F_{12}^{-1} \cdot F_{21} = R_{12}^{-1}. \tag{2.6}$$

- (c) The converse implication is true at least for finite-dimensional semi-simple Hopf algebras [9]. Up to complications due to the fact that the Yangian $Y(\mathfrak{sl}_2)$ has only a pseudo-universal R -matrix, the results in [7] and in this paper suggest that it applies to $Y(\mathfrak{sl}_2)$ as well.
- (d) If F is a factorizing twist, then the opposite coproduct can be made cocommutative using the twist F_{21} rather than F_{12} :

$$F_{21} \cdot \Delta^{\text{op}} X \cdot F_{21}^{-1} = F_{12} \cdot \Delta X \cdot F_{12}^{-1}. \tag{2.7}$$

- (e) Let $\gamma \in \mathcal{A}$ be invertible and $\varepsilon(\gamma) = 1$, then

$$F_\gamma = (\gamma \otimes \gamma) \cdot F \cdot (\Delta\gamma)^{-1} \tag{2.8}$$

is another co-unital 2-cocycle. Furthermore, the Hopf algebras \mathcal{A}_F and \mathcal{A}_{F_γ} obtained by twisting with F , respectively, F_γ are isomorphic under an inner automorphism. The twists F and F_γ are called cohomologous in this case. These ‘gauge’ degrees of freedom can be understood in terms of non-Abelian cohomology (see, e.g., [12]).

2.2. Notation and conventions

In this section we explain our notation for representations of the Lie algebra \mathfrak{sl}_2 and for evaluation representations of the Yangian $Y(\mathfrak{sl}_2)$.

2.2.1. Representations of the \mathfrak{sl}_2 Lie algebra. We choose a Cartan–Weyl basis (H, E, F) for the Lie algebra \mathfrak{sl}_2 ,

$$[H, E] = E \quad [H, F] = -F \quad [E, F] = 2H. \tag{2.9}$$

Let λ denote the highest weight of the representation V_λ , $|0\rangle$ be the highest-weight vector and $(|0\rangle, |1\rangle, \dots)$ a weight basis such that

$$H|k\rangle = (\lambda - k)|k\rangle \quad k \in \mathbb{N}_0 \tag{2.10}$$

i.e. we have finite-dimensional representations of dimension $2\lambda + 1$ for $2\lambda \in \mathbb{N}_0$. A representation for which the eigenvalue of the Casimir element $C^{(2)} = H^2 + \frac{1}{2}(EF + FE)$ equals $\lambda(\lambda + 1)$, is given by

$$E|k\rangle = e_k|k - 1\rangle \tag{2.11a}$$

$$F|k\rangle = f_k|k + 1\rangle \tag{2.11b}$$

where $e_k = k$ and $f_k = 2\lambda - k$.

In the following, we use the same notation for finite- and infinite-dimensional highest-weight representations. This means that in the finite-dimensional case the sequence of basis

vectors $(|0\rangle, |1\rangle, \dots)$ terminates, and, for example, in the case of dimension $2\lambda + 1$, $2\lambda \in \mathbb{N}_0$, the vectors $|k\rangle$ for $k > 2\lambda$ can be omitted or set to zero in all formulae.

Likewise, for the tensor product $V_{\lambda_1} \otimes V_{\lambda_2}$, the subspace of weight $\lambda_1 + \lambda_2 - m$ is spanned by $(|m\rangle \otimes |0\rangle, |m-1\rangle \otimes |1\rangle, \dots, |0\rangle \otimes |m\rangle)$. In the finite-dimensional case we have $|\ell\rangle \otimes |k\rangle = 0$ if $\ell > 2\lambda_1$ or $k > 2\lambda_2$, i.e. a number of vectors can be omitted from the generating set. In the following, we arrange all statements in such a way that this does not change the formulae.

2.2.2. *The Yangian $Y(\mathfrak{sl}_2)$.* The Yangian $Y(\mathfrak{sl}_2)$ is the free associative algebra over \mathbb{C} generated by $t_{ij}^{(r)}$, $i, j \in \{1, 2\}$, $r \in \mathbb{N}_0$, modulo the ideal generated by the relations

$$R_{k\ell}^{ij}(u-v) T_m^k(u) T_n^\ell(v) = T_\ell^j(v) T_k^i(u) R_{mn}^{k\ell}(u-v) \tag{2.12}$$

$$\text{qdet } T(u) = 1 \tag{2.13}$$

for all $i, j, m, n \in \{1, 2\}$, where summation over k and ℓ is understood. Here $R_{k\ell}^{ij}(u) = \delta_\ell^i \delta_k^j + \delta_k^i \delta_\ell^j 1/u$, and the power series

$$T_j^i(u) = \delta_{ij} + \sum_{r=1}^{\infty} t_{ij}^{(r)} u^{-r} \tag{2.14}$$

are to be expanded. The quantum determinant is defined as

$$\text{qdet } T(u) = T_1^1(u + \frac{1}{2}) T_2^2(u - \frac{1}{2}) - T_2^1(u + \frac{1}{2}) T_1^2(u - \frac{1}{2}). \tag{2.15}$$

2.2.3. *Evaluation representations of $Y(\mathfrak{sl}_2)$.* For a given representation V_λ of the Lie algebra \mathfrak{sl}_2 , we obtain an evaluation representation $V_\lambda(\delta)$ of $Y(\mathfrak{sl}_2)$ setting

$$\begin{aligned} A(u) &= T_1^1(u) = u - \delta + \eta H & B(u) &= T_2^1(u) = \eta F \\ C(u) &= T_1^2(u) = \eta E & D(u) &= T_2^2(u) = u - \delta - \eta H \end{aligned} \tag{2.16}$$

for the functionals in (2.14). Here the parameter δ is associated with the representation, η is a constant which could be eliminated by a rescaling, but is part of the common notation. In particular, equation (2.16) involves a homomorphism of algebras $Y(\mathfrak{sl}_2) \rightarrow U(\mathfrak{sl}_2)$. In (2.16) we have chosen a normalization different from the usual one. This will make the calculations easier, but will not affect the structure of the results. The quantum determinant, however, is now given by

$$\begin{aligned} \text{qdet } T(u) &= A(u + \frac{1}{2}\eta) D(u - \frac{1}{2}\eta) - B(u + \frac{1}{2}\eta) C(u - \frac{1}{2}\eta) \\ &= (u - \delta + \frac{1}{2}\eta)(u - \delta - \frac{1}{2}\eta) - \eta^2 \lambda(\lambda + 1) \end{aligned} \tag{2.17}$$

so strictly speaking we make calculations in $V_\lambda(\delta)$ as in a representation of $Y(\mathfrak{gl}_2)$. For the action of the operators defined in (2.16) on $V_{\lambda_j}(\delta_j)$ we write

$$A(u)|k\rangle = (u - \delta_j + \eta(\lambda_j - k))|k\rangle =: a_k^{(j)}(u)|k\rangle \tag{2.18a}$$

$$B(u)|k\rangle = \eta f_k^{(j)}|k+1\rangle \tag{2.18b}$$

$$C(u)|k\rangle = \eta e_k^{(j)}|k-1\rangle \tag{2.18c}$$

$$D(u)|k\rangle = (u - \delta_j - \eta(\lambda_j - k))|k\rangle =: d_k^{(j)}(u)|k\rangle \tag{2.18d}$$

where $e_k^{(j)} f_{k-1}^{(j)} = k(2\lambda_j - k + 1)$.

The coproduct in $Y(\mathfrak{sl}_2)$ can be written as

$$\begin{aligned} \Delta A(u) &= A(u) \otimes A(u) + C(u) \otimes B(u) \\ \Delta B(u) &= B(u) \otimes A(u) + D(u) \otimes B(u) \\ \Delta C(u) &= A(u) \otimes C(u) + C(u) \otimes D(u) \\ \Delta D(u) &= B(u) \otimes C(u) + D(u) \otimes D(u). \end{aligned} \tag{2.19}$$

It might seem more natural to call this Δ^{op} rather than Δ since it is the opposite of the usual matrix product, but our results look more natural if this coproduct is used. It is the same as defined in (2.14) of [7].

The coproduct then acts on $V_{\lambda_1}(\delta_1) \otimes V_{\lambda_2}(\delta_2)$ as

$$\Delta A(u)|\ell, k\rangle = a_\ell^{(1)}(u) a_k^{(2)}(u) |\ell, k\rangle + \eta^2 e_\ell^{(1)} f_k^{(2)} |\ell - 1, k + 1\rangle \tag{2.20a}$$

$$\Delta B(u)|\ell, k\rangle = \eta f_\ell^{(1)} a_k^{(2)}(u) |\ell + 1, k\rangle + \eta d_\ell^{(1)}(u) f_k^{(2)} |\ell, k + 1\rangle \tag{2.20b}$$

$$\Delta C(u)|\ell, k\rangle = \eta a_\ell^{(1)}(u) e_k^{(2)} |\ell, k - 1\rangle + \eta e_\ell^{(1)} d_k^{(2)}(u) |\ell - 1, k\rangle \tag{2.20c}$$

$$\Delta D(u)|\ell, k\rangle = \eta^2 f_\ell^{(1)} e_k^{(2)} |\ell + 1, k - 1\rangle + d_\ell^{(1)}(u) d_k^{(2)}(u) |\ell, k\rangle \tag{2.20d}$$

where $|\ell, k\rangle := |\ell\rangle \otimes |k\rangle$.

2.3. Irreducibility of evaluation representations

When we study the conditions for the existence of the factorizing twists, we refer to the corresponding conditions for the existence of R -matrices. Here we state a few results which are due to Tarasov and can be found in [13, 14], for example.

Theorem 2.6 (Tarasov). *Each finite-dimensional irreducible representation of the Yangian $Y(\mathfrak{sl}_2)$ over \mathbb{C} is isomorphic to a tensor product of evaluation representations. Two such tensor products describe isomorphic representations if and only if they are related by a permutation of the tensor factors.*

Theorem 2.7 (Tarasov). *The tensor product of finite-dimensional evaluation representations $V_{\lambda_1}(\delta_1) \otimes V_{\lambda_2}(\delta_2)$ is reducible if and only if*

$$\pm(\delta_1 - \delta_2)/\eta = \lambda_1 + \lambda_2 - j + 1 \tag{2.21}$$

for an integer j satisfying $0 < j \leq \min\{2\lambda_1, 2\lambda_2\}$. In this case the representation is not completely reducible nor isomorphic to $V_{\lambda_2}(\delta_2) \otimes V_{\lambda_1}(\delta_1)$.

Remark 2.8. *The existence of these representations is closely related to the failure of the Yangian to be quasi-triangular. In particular, the condition (2.1c) is not satisfied in this case. In evaluation representations the R -matrix is given by a matrix $R(\delta_1 - \delta_2)$ of rational functions of $\delta_1 - \delta_2$. This matrix is fixed by (2.1c) which reads here*

$$\Delta^{\text{op}} X(u) = R(\delta_1 - \delta_2) \Delta X(u) R(\delta_1 - \delta_2)^{-1} \tag{2.22}$$

but has a pole or fails to be invertible at the ‘singularities’ (2.21). The simplest example is the case $V_{1/2}(\delta_1) \otimes V_{1/2}(\delta_2)$, where

$$R(\delta_1 - \delta_2) = \frac{\delta_1 - \delta_2}{\delta_1 - \delta_2 + \eta} \mathbb{1} + \frac{\eta}{\delta_1 - \delta_2 + \eta} P \tag{2.23}$$

where $P(a \otimes b) = b \otimes a$. In the case of $\delta_1 - \delta_2 = \pm\eta$, either R or R^{-1} do not exist.

3. The triangular part of the factorizing twist

The essence of the modification of the functional Bethe ansatz used in [7] in order to construct factorizing twists is the diagonalization of the operator $\Delta D(u)$ on $V_{\lambda_1}(\delta_1) \otimes V_{\lambda_2}(\delta_2)$ (one could equally well have used $\Delta A(u)$).

In the following, we will not use a modified functional Bethe ansatz, but rather perform the necessary change of basis explicitly. We find that the prescription to diagonalize $\Delta D(u)$ fixes the triangular part of the factorizing twist. Diagonalizing $\Delta A(u)$ likewise results in the triangular part of (another) factorizing twist. However, it is not possible to diagonalize both $\Delta A(u)$ and $\Delta D(u)$ simultaneously in a way that is independent of u .

3.1. Eigenvectors of some special matrices

A few technical results are needed to accomplish the diagonalization. The following statements are all elementary and straightforward to prove.

Lemma 3.1. *The eigenvectors of the $(n + 1) \times (n + 1)$ real or complex matrix*

$$M = \begin{pmatrix} c_0 & b_0 & & & & \\ & c_1 & b_1 & & & \\ & & \ddots & \ddots & & \\ & & & c_{n-1} & b_{n-1} & \\ & & & & & c_n \end{pmatrix} \tag{3.1}$$

with pairwise distinct c_i are the column vectors $v^{(k)}$, $0 \leq k \leq n$, with the components

$$v_j^{(k)} = \begin{cases} 0 & \text{if } j > k \\ \prod_{i=j}^{k-1} \frac{b_i}{c_k - c_i} & \text{if } j \leq k \end{cases} \tag{3.2}$$

$0 \leq j \leq n$, where the empty product is by definition equal to 1. The eigenvalue of $v^{(k)}$ is c_k .

Corollary 3.2. *The matrix M in (3.1) can be diagonalized using*

$$S^{-1} \cdot M \cdot S = \text{diag}(c_0, \dots, c_n) \tag{3.3}$$

where the change of basis is given by the (upper-triangular) matrix with coefficients

$$S_{jk} = v_j^{(k)} \tag{3.4}$$

$0 \leq j, k \leq n$, where the first index denotes the row and the second index denotes the column of the matrix.

Lemma 3.3. *For pairwise distinct c_1, \dots, c_m the following sum vanishes:*

$$\sum_{i=1}^m \prod_{\substack{j=1 \\ j \neq i}}^m \frac{1}{c_j - c_i} = 0. \tag{3.5}$$

The proof is again elementary. An elegant argument can be found in [15], for example. This identity is necessary to find the inverse matrix:

Corollary 3.4. *The inverse matrix of S in (3.4) is given by*

$$S_{jk}^{-1} = \begin{cases} 0 & \text{if } j > k \\ \prod_{i=j+1}^k \frac{b_{i-1}}{c_j - c_i} & \text{if } j \leq k \end{cases} \tag{3.6}$$

$0 \leq j, k \leq n$. It is again upper triangular.

We will also need analogous statements referring to the transposed matrix:

Corollary 3.5. *The transposed matrix $\tilde{M} = M^t$ is diagonalized by*

$$\tilde{S}^{-1} \tilde{M} \tilde{S} = \text{diag}(c_0, \dots, c_n) \tag{3.7}$$

where $\tilde{S} = (S^{-1})^t$ is now lower triangular.

3.2. Diagonalizing $\Delta D(u)$

The matrix of $\Delta D(u)$ on $V_{\lambda_1}(\delta_1) \otimes V_{\lambda_2}(\delta_2)$ decomposes into blocks acting on invariant subspaces V_m which are spanned by

$$V_m = \text{span}\{|m, 0\rangle, |m - 1, 1\rangle, \dots, |0, m\rangle\} \tag{3.8}$$

$m \in \mathbb{N}_0$. These subspaces are always finite dimensional even if the representations under study are infinite dimensional. In that case there are just infinitely many blocks of growing but finite size.

On each V_m , $\Delta D(u)$ has the form

$$\Delta D(u) = \begin{pmatrix} c_0 & & & & & \\ b_0 & c_1 & & & & \\ & b_1 & \ddots & & & \\ & & \ddots & c_{m-1} & & \\ & & & b_{m-1} & c_m & \end{pmatrix} \tag{3.9}$$

where the matrix coefficients belong to $|0, m\rangle, |1, m - 1\rangle, \dots, |m, 0\rangle$ from top to bottom and left to right. We find using (2.20)

$$c_k = d_k^{(1)}(u) \cdot d_{m-k}^{(2)}(u) \tag{3.10a}$$

$$b_k = \eta^2 f_k^{(1)} e_{m-k}^{(2)} \tag{3.10b}$$

$$c_\ell - c_k = \eta(\ell - k) g(m - k - \ell) \tag{3.10c}$$

where $g(x) := \delta_1 - \delta_2 + \eta(\lambda_1 - \lambda_2 + x)$. Therefore, each of the blocks of $\Delta D(u)$ can be diagonalized using corollary 3.5.

Proposition 3.6. *We label the eigenvectors $v_{\ell k}$ of $\Delta D(u)$ by two indices, $k, \ell \in \mathbb{N}_0$. These eigenvectors are given by*

$$\begin{aligned} v_{\ell k} &= q_{\ell k} \sum_{j=0}^{k+\ell} \tilde{S}_{j\ell} |j, k + \ell - j\rangle \\ &= q_{\ell k} \sum_{n=0}^k \prod_{i=1}^n \frac{b_{\ell+i-1}}{c_\ell - c_{j+\ell}} | \ell + n, k - n \rangle \\ &= q_{\ell k} \sum_{n=0}^k M_n^{k,\ell} | \ell + n, k - n \rangle \end{aligned} \tag{3.11}$$

for some coefficients $q_{\ell k}$. We have defined

$$M_n^{k,\ell} := \frac{(-\eta)^n}{n!} \prod_{j=1}^n \frac{f_{\ell+j-1}^{(1)} e_{k-j+1}^{(2)}}{g(k-\ell-j)}. \quad (3.12)$$

The eigenvalue of $v_{\ell k}$ is $d_\ell^{(1)}(u) \cdot d_k^{(2)}(u)$. In particular, the eigenvectors do not depend on u .

Here the coefficients $q_{\ell k}$ reflect the freedom in normalizing the eigenvectors. They will be fixed using other conditions in section 4. It is amazing that this change of basis already contains full information concerning one of the two triangular parts of the Gauss decomposition of the R -matrix on a generic evaluation representation. This fact will become clear in section 5. Similarly, we find

Proposition 3.7. *The inverse transformation of (3.11) is given by*

$$|\ell, k\rangle = \sum_{n=0}^k \frac{\eta^n}{n!} \prod_{j=1}^n \frac{f_{\ell+j-1}^{(1)} e_{k-j+1}^{(2)}}{g((k-n) - (\ell+n) + j)} q_{\ell+n, k-n}^{-1} v_{\ell+n, k-n}. \quad (3.13)$$

We will need the following properties of the coefficients $M_n^{k,\ell}$ later:

Lemma 3.8. *The matrix elements $M_n^{k,\ell}$ defined in (3.12) satisfy*

$$M_0^{k,\ell} = 1 \quad (3.14a)$$

$$\frac{M_{n+1}^{k,\ell}}{M_n^{k,\ell}} = -\frac{\eta f_{\ell+n}^{(1)} e_{k-n}^{(2)}}{(n+1)g(k-\ell-n-1)} \quad (3.14b)$$

$$\frac{M_n^{k,\ell-1}}{M_n^{k,\ell}} = \frac{f_{\ell-1}^{(1)} g(k-\ell-n)}{f_{\ell+n-1}^{(1)} g(k-\ell)} \quad (3.14c)$$

$$\frac{M_n^{k-1,\ell}}{M_n^{k,\ell}} = \frac{e_{k-n}^{(2)} g(k-\ell-1)}{e_k^{(2)} g(k-\ell-n-1)}. \quad (3.14d)$$

3.3. Diagonalizing $\Delta A(u)$

Likewise, it is possible to diagonalize $\Delta A(u)$. Again its matrix decomposes into blocks on the subspaces V_m which are of the form

$$\Delta A(u) = \begin{pmatrix} c_0 & b_0 & & & & \\ & c_1 & b_1 & & & \\ & & & \ddots & & \\ & & & & c_{m-1} & b_{m-1} \\ & & & & & c_m \end{pmatrix}. \quad (3.15)$$

Here

$$c_k = a_k^{(1)}(u) \cdot a_{m-k}^{(2)}(u) \quad (3.16a)$$

$$b_k = \eta^2 e_{k+1}^{(1)} f_{m-k-1}^{(2)} \quad (3.16b)$$

$$c_\ell - c_k = \eta(\ell - k) \tilde{g}(m - k - \ell) \quad (3.16c)$$

where now $\tilde{g}(x) := \delta_2 - \delta_1 + \eta(\lambda_1 - \lambda_2 + x)$, i.e. compared with $g(x)$ in (3.10c) we just exchange $\delta_1 \leftrightarrow \delta_2$.

Each of the blocks of $\Delta A(u)$ can thus be diagonalized using lemma 3.1.

Proposition 3.9. *The eigenvectors of $\Delta A(u)$ on $V_{\lambda_1}(\delta_2) \otimes V_{\lambda_2}(\delta_2)$ are given by $\tilde{v}_{\ell k}$, $k, \ell \in \mathbb{N}_0$,*

$$\begin{aligned} \tilde{v}_{\ell k} &= \tilde{q}_{\ell k} \sum_{j=0}^{k+\ell} S_{j\ell} |j, k + \ell - j\rangle \\ &= \tilde{q}_{\ell k} \sum_{n=0}^{\ell} \tilde{M}_n^{k,\ell} |\ell - n, k + n\rangle \end{aligned} \tag{3.17}$$

where we have abbreviated

$$\tilde{M}_n^{k,\ell} := \frac{\eta^n}{n!} \prod_{j=1}^n \frac{e_{\ell-j+1}^{(1)} f_{k+j-1}^{(2)}}{\tilde{g}(k - \ell + j)}. \tag{3.18}$$

The eigenvalue of $\tilde{v}_{\ell k}$ is $a_{\ell}^{(1)}(u) a_k^{(2)}(u)$.

Similarly, we find

Proposition 3.10. *The inverse transformation of (3.17) is given by*

$$|\ell, k\rangle = \sum_{n=0}^{\ell} \frac{(-\eta)^n}{n!} \prod_{j=1}^n \frac{e_{\ell-j+1}^{(1)} f_{k+j-1}^{(2)}}{\tilde{g}((k+n) - (\ell-n) - j)} \tilde{q}_{\ell-n, k+n}^{-1} \tilde{v}_{\ell-n, k+n}. \tag{3.19}$$

The following properties of the coefficients $\tilde{M}_n^{k,\ell}$ will later be useful:

Lemma 3.11. *The matrix elements $\tilde{M}_n^{k,\ell}$ defined in (3.18) satisfy*

$$\tilde{M}_0^{k,\ell} = 1 \tag{3.20a}$$

$$\frac{\tilde{M}_{n+1}^{k,\ell}}{\tilde{M}_n^{k,\ell}} = \frac{\eta e_{\ell-n}^{(1)} f_{k+n}^{(2)}}{(n+1) \tilde{g}(k - \ell + n + 1)} \tag{3.20b}$$

$$\frac{\tilde{M}_n^{k,\ell-1}}{\tilde{M}_n^{k,\ell}} = \frac{e_{\ell-n}^{(1)} \tilde{g}(k - \ell + 1)}{e_{\ell}^{(1)} \tilde{g}(k - \ell + n + 1)} \tag{3.20c}$$

$$\frac{\tilde{M}_n^{k-1,\ell}}{\tilde{M}_n^{k,\ell}} = \frac{f_{k-1}^{(2)} \tilde{g}(k - \ell + n)}{f_{k+n-1}^{(2)} \tilde{g}(k - \ell)}. \tag{3.20d}$$

3.4. Towards a universal expression

In order to derive a form of the change of basis (3.11) diagonalizing $\Delta D(u)$ which does not depend on the particular representation, it is necessary to express the basis vectors $|\ell + n, k - n\rangle$ in terms of $|\ell, k\rangle$. Using

$$F^n \otimes E^n |\ell, k\rangle = \left(\prod_{j=1}^n f_{\ell+j-1}^{(1)} e_{k-j+1}^{(2)} \right) |\ell + n, k - n\rangle \tag{3.21}$$

and

$$H \otimes \mathbb{1} |\ell, k\rangle = (\lambda_1 - \ell) |\ell, k\rangle \quad \mathbb{1} \otimes H |\ell, k\rangle = (\lambda_2 - k) |\ell, k\rangle \tag{3.22}$$

equation (3.11) reads

$$v_{\ell k} = q_{\ell k} \sum_{n=0}^k \frac{(-\eta)^n}{n!} F^n \otimes E^n \left(\prod_{j=1}^n (\delta_1 - \delta_2 + \eta (H \otimes \mathbb{1} - \mathbb{1} \otimes H) - \eta j)^{-1} \right) |\ell, k\rangle. \tag{3.23}$$

It is apparent that in order to generalize this to generic highest-weight representations, it is sufficient to make the sum infinite. We conclude:

Proposition 3.12. *The expression*

$$F_{12}^{-1} = \left(\sum_{n=0}^{\infty} \frac{(-\eta)^n}{n!} F^n \otimes E^n \prod_{j=1}^n (\delta_1 - \delta_2 + \eta (H \otimes \mathbb{1} - \mathbb{1} \otimes H) - \eta j)^{-1} \right) Q_{12}^{-1} \quad (3.24)$$

where we denote

$$Q_{12}^{-1} |\ell, k\rangle = q_{\ell k} |\ell, k\rangle \quad (3.25)$$

specializes to the change of basis (3.23) and (3.11) on all representations $V_{\lambda_1}(\delta_1) \otimes V_{\lambda_2}(\delta_2)$. Recall that this change of basis diagonalizes $\Delta D(u)$.

Remark 3.13.

- (a) The result (3.24) agrees with (4.53) in [7].
- (b) The operator $F_{12}^{-1} Q_{12}$ decomposes into (possibly infinitely many) triangular matrices on each representation. Its diagonal elements are given by the $n = 0$ summand, i.e. all its coefficients on the diagonal equal 1. This is the triangular part of the Gauss decomposition of the twist, while Q_{12}^{-1} describes its diagonal part.
- (c) The question of under what conditions expression (3.24) is well defined is discussed in section 5.4.

Similarly, we can obtain the inverse transformation starting with (3.13).

Proposition 3.14. *The expression*

$$F_{12} = Q_{12} \sum_{n=0}^{\infty} \frac{\eta^n}{n!} \left(\prod_{j=1}^n (\delta_1 - \delta_2 + \eta (H \otimes \mathbb{1} - \mathbb{1} \otimes H) + \eta j)^{-1} \right) F^n \otimes E^n \quad (3.26)$$

specializes to the inverse change of basis (3.13) for the diagonalization of $\Delta D(u)$ on all representations $V_{\lambda_1}(\delta_1) \otimes V_{\lambda_2}(\delta_2)$.

Analogous results can be obtained for the case in which $\Delta A(u)$ rather than $\Delta D(u)$ is diagonalized.

Proposition 3.15. *The expressions*

$$\tilde{F}_{12}^{-1} = \left(\sum_{n=0}^{\infty} \frac{\eta^n}{n!} E^n \otimes F^n \prod_{j=1}^n (\delta_2 - \delta_1 + \eta (H \otimes \mathbb{1} - \mathbb{1} \otimes H) + \eta j)^{-1} \right) \tilde{Q}_{12}^{-1} \quad (3.27)$$

$$\tilde{F}_{12} = \tilde{Q}_{12} \sum_{n=0}^{\infty} \frac{(-\eta)^n}{n!} \left(\prod_{j=1}^n (\delta_2 - \delta_1 + \eta (H \otimes \mathbb{1} - \mathbb{1} \otimes H) - \eta j)^{-1} \right) E^n \otimes F^n \quad (3.28)$$

specialize to the change of basis (3.17) respectively its inverse transformation (3.19) on all representations $V_{\lambda_1}(\delta_1) \otimes V_{\lambda_2}(\delta_2)$. Recall that this change of basis diagonalizes $\Delta A(u)$. Here we write

$$\tilde{Q}_{12}^{-1} |\ell, k\rangle = \tilde{q}_{\ell k} |\ell, k\rangle. \quad (3.29)$$

4. The diagonal part of the factorizing twist

In this section we study the conditions that restrict the $q_{\ell k}$ coefficients of the diagonal part of the twist.

4.1. *Cocommutativity*

In order to make F_{12} in (3.24) a factorizing twist, it is of course not sufficient to diagonalize $\Delta D(u)$. The desired property is that the twisted Yangian coproduct $F\Delta X(u)F^{-1}$ should be cocommutative. It is thus necessary to fix the coefficients $q_{\ell k}$ of Q_{12}^{-1} appropriately.

A case-by-case analysis of the action of, for example, $\Delta B(u)$ in the new basis $v_{\ell k}$, see (3.23), in special evaluation representations shows that $F \cdot \Delta B(u) \cdot F^{-1}$ is of the form

$$\Delta B(u)v_{\ell k} = \alpha_{\ell k}(u) \cdot \frac{q_{\ell k}}{q_{\ell+1,k}} v_{\ell+1,k} + \beta_{\ell k}(u) \cdot \frac{q_{\ell k}}{q_{\ell,k+1}} v_{\ell,k+1} \tag{4.1}$$

where $\alpha_{\ell k}(u)$ and $\beta_{\ell k}(u)$ are as yet unspecified functions of u . Cocommutativity means that the expression for $F \cdot \Delta B(u) \cdot F^{-1}$ written in $U(\mathfrak{sl}_2)$ is symmetric, i.e. that

$$\alpha_{\ell k}(u) \cdot \frac{q_{\ell k}}{q_{\ell+1,k}} = \left(\beta_{k\ell}(u) \cdot \frac{q_{k\ell}}{q_{k+1,\ell}} \right)_{1 \leftrightarrow 2} \tag{4.2}$$

where $1 \leftrightarrow 2$ indicates that $\lambda_1 \leftrightarrow \lambda_2$ and $\delta_1 \leftrightarrow \delta_2$ have to be exchanged in the corresponding expressions.

It is apparent that there is a freedom to choose a multiplicative factor in the coefficients $q_{\ell k}$ which depends only on $k + \ell$ and which is symmetric under the exchange $1 \leftrightarrow 2$:

$$q_{\ell k} \mapsto q_{\ell k} \cdot r_{k+\ell} \quad r_{k+\ell} = (r_{k+\ell})_{1 \leftrightarrow 2}. \tag{4.3}$$

Since we work only in representations, it is not easy to see whether this freedom (which is also mentioned in [2, 6]) exhausts the full freedom of choosing a cohomologous twist (see remark 2.5).

Finally, it is possible to find a solution for the quotients $q_{\ell k}/q_{\ell+1,k}$, etc which makes the new coproduct $F \cdot \Delta B(u) \cdot F^{-1}$ cocommutative. We show that in this case the coproduct on the other generators is cocommutative as well. The following proposition seems technically very complicated, but it is just a reasonably straightforward abstraction from explicit calculations in particular finite-dimensional representations.

Proposition 4.1. *If the coefficients $q_{\ell k}$ of Q_{12}^{-1} satisfy the recursion relations*

$$\frac{q_{\ell k}}{q_{\ell-1,k}} = \frac{g(-\ell)}{g(k-\ell)} \quad \frac{q_{\ell,k+1}}{q_{\ell k}} = \frac{g(k-\ell)}{g(-2\lambda_1+k)} \tag{4.4}$$

where $g(x) = \delta_1 - \delta_2 + \eta(\lambda_1 - \lambda_2 + x)$, then the action of the coproducts on the basis vectors $v_{\ell k}$, see (3.23), is given by

$$\Delta D(u)v_{\ell k} = d_\ell^{(1)}(u) \cdot d_k^{(2)}(u)v_{\ell k} \tag{4.5}$$

$$\Delta B(u)v_{\ell k} = \eta f_\ell^{(1)} d_k^{(2)}(u) \frac{g(2\lambda_2 - \ell)}{g(k - \ell)} v_{\ell+1,k} + \eta d_\ell^{(1)}(u) f_k^{(2)} \frac{g(-2\lambda_1 + k)}{g(k - \ell)} v_{\ell,k+1} \tag{4.6}$$

$$\Delta C(u)v_{\ell k} = \eta e_\ell^{(1)} d_k^{(2)}(u) \frac{g(-\ell)}{g(k - \ell)} v_{\ell-1,k} + \eta d_\ell^{(1)}(u) e_k^{(2)} \frac{g(k)}{g(k - \ell)} v_{\ell,k-1}. \tag{4.7}$$

Proof. Diagonality of $\Delta D(u)$ was stated in proposition 3.6. For $\Delta B(u)$ we first calculate

$$\begin{aligned} \Delta B(u) v_{\ell k} &= q_{\ell k} \sum_{i=0}^k M_i^{k,\ell} \Delta B(u) |\ell + i, k - i\rangle \\ &= q_{\ell k} \sum_{i=0}^k M_i^{k,\ell} \eta (d_{\ell+i}^{(1)}(u) f_{k-i}^{(2)} |\ell + i, k - i + 1\rangle + f_{\ell+i}^{(1)} a_{k-i}^{(2)}(u) |\ell + i + 1, k - i\rangle) \\ &= q_{\ell k} \sum_{i=0}^k M_i^{k,\ell} \eta f_{\ell+i}^{(1)} \left(a_{k-i}^{(2)}(u) - d_{\ell+i+1}^{(1)}(u) \frac{\eta e_{k-i}^{(2)} f_{k-i-1}^{(2)}}{(i+1)g(k-\ell-i-1)} \right) \\ &\quad \times |\ell + i + 1, k - i\rangle + q_{\ell k} \eta d_{\ell}^{(1)}(u) f_k^{(2)} |\ell, k + 1\rangle. \end{aligned} \tag{4.8}$$

Here we use the notation of (2.18). In the third line, the summation index was shifted and lemma 3.8 was used.

It remains to show that writing coefficients

$$c_{\ell k}^1 = \eta d_k^{(2)}(u) f_{\ell}^{(1)} \frac{g(2\lambda_2 - \ell)}{g(k - \ell)} \quad c_{\ell k}^2 = \eta d_{\ell}^{(1)}(u) f_k^{(2)} \frac{g(-2\lambda_1 + k)}{g(k - \ell)} \tag{4.9}$$

the coproduct $\Delta B(u)$ has the following form:

$$\begin{aligned} \Delta B(u) v_{\ell k} &= c_{\ell k}^1 v_{\ell+1,k} + c_{\ell k}^2 v_{\ell,k+1} \\ &= c_{\ell k}^1 q_{\ell+1,k} \sum_{i=0}^k M_i^{k,\ell+1} |\ell + i + 1, k - i\rangle + c_{\ell k}^2 q_{\ell,k+1} \sum_{i=0}^{k+1} M_i^{k+1,\ell} |\ell + i, k - i + 1\rangle \\ &= \sum_{i=0}^k M_i^{k,\ell} \eta f_{\ell+i}^{(1)} \left(c_{\ell k}^1 q_{\ell+1,k} \frac{g(k - \ell - 1)}{\eta f_{\ell}^{(1)} g(k - \ell - i - 1)} \right. \\ &\quad \left. - c_{\ell k}^2 q_{\ell,k+1} \frac{e_{k+1}^{(2)}}{(i+1)g(k - \ell)} \right) |\ell + i + 1, k - i\rangle + c_{\ell k}^2 q_{\ell,k+1} |\ell, k + 1\rangle. \end{aligned} \tag{4.10}$$

In the last line, the summation index was shifted and, using lemma 3.8 several times, the coefficient $M_i^{k,\ell}$ was cast in the same form as in (4.8). Both expressions (4.8) and (4.10) are equal if and only if for all $k, \ell \in \mathbb{N}_0$

$$q_{\ell k} \eta f_k^{(2)} d_{\ell}^{(1)}(u) = c_{\ell k}^2 q_{\ell,k+1} \tag{4.11}$$

and for all $i \in \{0, \dots, k\}$

$$\begin{aligned} a_{k-i}^{(2)}(u) - d_{\ell+i+1}^{(1)}(u) \frac{\eta e_{k-i}^{(2)} f_{k-i-1}^{(2)}}{(i+1)g(k-\ell-i-1)} \\ = c_{\ell k}^1 \cdot \frac{q_{\ell+1,k}}{q_{\ell k}} \frac{g(k-\ell-1)}{\eta f_{\ell}^{(1)} g(k-\ell-i-1)} - c_{\ell k}^2 \frac{q_{\ell,k+1}}{q_{\ell k}} \cdot \frac{e_{k+1}^{(2)}}{(i+1)g(k-\ell)}. \end{aligned} \tag{4.12}$$

The first condition holds since the recursion relations (4.4) are satisfied. The second simplifies to

$$\begin{aligned} a_{k-i}^{(2)}(u) - d_{\ell+i+1}^{(1)}(u) \frac{\eta e_{k-i}^{(2)} f_{k-i-1}^{(2)}}{(i+1)g(k-\ell-i-1)} \\ = d_k^{(2)}(u) \frac{g(2\lambda_2 - \ell)g(-\ell - 1)}{g(k - \ell)g(k - \ell - i - 1)} - d_{\ell}^{(1)}(u) \frac{\eta e_{k+1}^{(2)} f_k^{(2)}}{(i+1)g(k - \ell)} \end{aligned} \tag{4.13}$$

which can be verified in a direct computation.

The procedure to prove the assertion for $\Delta C(u)$ is similar. In order to understand in detail how cocommutativity arises from the choice of the coefficients $q_{\ell k}$ in (4.4), it is important to contrast the calculation for $\Delta B(u)$ with that for $\Delta C(u)$. Thus we give the main steps in detail again. Firstly,

$$\begin{aligned} \Delta C(u) v_{\ell k} &= q_{\ell k} \sum_{i=0}^k M_i^{k,\ell} \Delta C(u) |\ell + i, k - i\rangle \\ &= q_{\ell k} \sum_{i=0}^k M_i^{k,\ell} \eta (a_{\ell+i}^{(1)}(u) e_{k-i}^{(2)} |\ell + i, k - i - 1\rangle + e_{\ell+i}^{(1)} d_{k-i}^{(2)}(u) |\ell + i - 1, k - i\rangle) \\ &= q_{\ell k} \sum_{i=1}^k M_i^{k,\ell} \left(d_{k-i}^{(2)}(u) \eta e_{\ell+i}^{(1)} - a_{\ell+i-1}^{(1)}(u) \frac{\text{ig}(k - \ell - i)}{f_{\ell+i-1}^{(1)}} \right) |\ell + i - 1, k - i\rangle \\ &\quad + q_{\ell k} \eta e_{\ell}^{(1)} d_k^{(2)}(u) |\ell - 1, k\rangle \end{aligned} \quad (4.14)$$

again using the notation of (2.18). In the third line, the summation index was shifted and lemma 3.8 used. Writing

$$c_{\ell k}^3 = \eta d_k^{(2)}(u) e_{\ell}^{(1)} \frac{g(-\ell)}{g(k - \ell)} \quad c_{\ell k}^4 = \eta d_{\ell}^{(1)}(u) e_k^{(2)} \frac{g(k)}{g(k - \ell)} \quad (4.15)$$

it remains to show that

$$\begin{aligned} \Delta C(u) v_{\ell k} &= c_{\ell k}^3 v_{\ell-1, k} + c_{\ell k}^4 v_{\ell, k-1} \\ &= c_{\ell k}^3 q_{\ell-1, k} \sum_{i=0}^k M_i^{k,\ell-1} |\ell + i - 1, k - i\rangle + c_{\ell k}^4 q_{\ell, k-1} \\ &\quad \times \sum_{i=0}^{k-1} M_i^{k-1,\ell} |\ell + i, k - i - 1\rangle \\ &= \sum_{i=1}^k M_i^{k,\ell} \left(c_{\ell k}^3 q_{\ell-1, k} \frac{f_{\ell-1}^{(1)} g(k - \ell - i)}{f_{\ell+i-1}^{(1)} g(k - \ell)} \right. \\ &\quad \left. - c_{\ell k}^4 q_{\ell, k-1} \frac{\text{ig}(k - \ell - 1)}{\eta e_k^{(2)} f_{\ell+i-1}^{(1)}} \right) |\ell + i - 1, k - i\rangle + c_{\ell k}^3 q_{\ell-1, k} |\ell - 1, k\rangle. \end{aligned} \quad (4.16)$$

In the last line, the summation index was shifted and, using lemma 3.8 several times, the coefficient $M_i^{k,\ell}$ was cast in the same form as in (4.14). Both expressions (4.14) and (4.16) are equal if and only if for all $k, \ell \in \mathbb{N}_0$

$$q_{\ell k} \eta e_{\ell}^{(1)} d_k^{(2)}(u) = c_{\ell k}^3 q_{\ell-1, k} \quad (4.17)$$

and for all $i \in \{0, \dots, k\}$

$$\begin{aligned} \eta d_{k-i}^{(2)}(u) e_{\ell+i}^{(1)} f_{\ell+i-1}^{(1)} - a_{\ell+i-1}^{(1)}(u) \text{ig}(k - \ell - i) \\ = c_{\ell k}^3 \frac{q_{\ell-1, k} f_{\ell-1}^{(1)} g(k - \ell - i)}{q_{\ell k} g(k - \ell)} - c_{\ell k}^4 \frac{q_{\ell, k-1} \text{ig}(k - \ell - 1)}{q_{\ell k} \eta e_k^{(2)}}. \end{aligned} \quad (4.18)$$

The first condition holds since the recursion relations (4.4) are satisfied. The second simplifies to

$$\begin{aligned} \eta d_{k-i}^{(2)}(u) e_{\ell+i}^{(1)} f_{\ell+i-1}^{(1)} - a_{\ell+i-1}^{(1)}(u) \operatorname{ig}(k - \ell - i) \\ = d_k^{(2)}(u) \frac{\eta e_\ell^{(1)} f_{\ell-1}^{(1)} g(k - \ell - i)}{g(k - \ell)} - d_\ell^{(1)}(u) \frac{\operatorname{ig}(k) g(-2\lambda_1 + k - 1)}{g(k - \ell)} \end{aligned} \quad (4.19)$$

which can be verified in a direct computation. \square

Writing the last proposition using operators, we obtain

Corollary 4.2. *The twisted coproduct on $V_{\lambda_1}(\delta_1) \otimes V_{\lambda_2}(\delta_2)$ can be expressed as follows:*

$$F \Delta D(u) F^{-1} = D(u) \otimes D(u) \quad (4.20a)$$

$$\begin{aligned} F \Delta B(u) F^{-1} = B(u) \otimes D(u) \frac{\delta_1 - \delta_2 + \eta(H \otimes \mathbb{1} + \lambda_2)}{\delta_1 - \delta_2 + \eta(H \otimes \mathbb{1} - \mathbb{1} \otimes H)} \\ + D(u) \otimes B(u) \frac{\delta_1 - \delta_2 + \eta(-\lambda_1 - \mathbb{1} \otimes H)}{\delta_1 - \delta_2 + \eta(H \otimes \mathbb{1} - \mathbb{1} \otimes H)} \end{aligned} \quad (4.20b)$$

$$\begin{aligned} F \Delta C(u) F^{-1} = C(u) \otimes D(u) \frac{\delta_1 - \delta_2 + \eta(H \otimes \mathbb{1} - \lambda_2)}{\delta_1 - \delta_2 + \eta(H \otimes \mathbb{1} - \mathbb{1} \otimes H)} \\ + D(u) \otimes C(u) \frac{\delta_1 - \delta_2 + \eta(\lambda_1 - \mathbb{1} \otimes H)}{\delta_1 - \delta_2 + \eta(H \otimes \mathbb{1} - \mathbb{1} \otimes H)}. \end{aligned} \quad (4.20c)$$

In particular, the twisted coproduct is cocommutative.

Remark 4.3.

- (a) The results for the new coproduct (4.20) are compared with the results of [7] in section 5.3.
- (b) Since the homomorphism of algebras $Y(\mathfrak{sl}_2) \rightarrow U(\mathfrak{sl}_2)$, which was used above to express the coproduct in terms of the Lie algebra generators E, F and H , is not a homomorphism of co-algebras, the above expressions for the new coproduct do not allow an easy conclusion concerning the new coproduct on the Yangian.
- (c) Cocommutativity of $F \Delta A(u) F^{-1}$ follows immediately because the relation for the quantum determinant (2.17) can be solved for $A(u)$. Since $\operatorname{qdet} T(u)$ is group-like, the relation extends to the coproduct, and since it is central, we can apply $F(\cdot)F^{-1}$ without changing the structure of the equation.

Finally, the analogous construction can be made for the twist \tilde{F}_{12} which is obtained if $\Delta A(u)$ is diagonalized. We just summarize the results.

Proposition 4.4. *If the coefficients $\tilde{q}_{\ell k}$ of \tilde{Q}_{12}^{-1} satisfy the recursion relations*

$$\frac{\tilde{q}_{\ell+1,k}}{\tilde{q}_{\ell k}} = \frac{\tilde{g}(k - \ell)}{\tilde{g}(2\lambda_2 - \ell)} \quad \frac{\tilde{q}_{\ell k}}{\tilde{q}_{\ell,k-1}} = \frac{\tilde{g}(k)}{\tilde{g}(k - \ell)} \quad (4.21)$$

where $\tilde{g}(x) = \delta_2 - \delta_1 + \eta(\lambda_1 - \lambda_2 + x)$, then the action of the coproducts on the basis vectors $\tilde{v}_{\ell k}$, see (3.17) and (3.27), is given by

$$\Delta A(u) \tilde{v}_{\ell k} = a_\ell^{(1)}(u) a_k^{(2)}(u) \tilde{v}_{\ell k} \quad (4.22)$$

$$\Delta B(u) \tilde{v}_{\ell k} = \eta f_\ell^{(1)} a_k^{(2)}(u) \frac{\tilde{g}(2\lambda_2 - \ell)}{\tilde{g}(k - \ell)} \tilde{v}_{\ell+1,k} + \eta f_k^{(2)} a_\ell^{(1)}(u) \frac{\tilde{g}(-2\lambda_1 + k)}{\tilde{g}(k - \ell)} \tilde{v}_{\ell,k+1} \quad (4.23)$$

$$\Delta C(u) \tilde{v}_{\ell k} = \eta e_\ell^{(1)} a_k^{(2)}(u) \frac{\tilde{g}(-\ell)}{\tilde{g}(k - \ell)} \tilde{v}_{\ell-1,k} + \eta e_k^{(2)} a_\ell^{(1)}(u) \frac{g(k)}{\tilde{g}(k - \ell)} \tilde{v}_{\ell,k-1}. \quad (4.24)$$

Proof. The proof is completely analogous to that of proposition 4.1. Here lemma 3.11 is used to deal with the coefficients $\widetilde{M}_i^{k,\ell}$ and with the shifts of the summation index which are necessary here. \square

Corollary 4.5. *The coproduct on $V_{\lambda_1}(\delta_1) \otimes V_{\lambda_2}(\delta_2)$ twisted with the twist \widetilde{F} diagonalizing $\Delta A(u)$ can be expressed as follows:*

$$\widetilde{F} \cdot \Delta A(u) \cdot \widetilde{F}^{-1} = A(u) \otimes A(u) \quad (4.25a)$$

$$\begin{aligned} \widetilde{F} \cdot \Delta B(u) \cdot \widetilde{F}^{-1} &= B(u) \otimes A(u) \frac{\delta_2 - \delta_1 + \eta (H \otimes \mathbb{1} + \lambda_2)}{\delta_2 - \delta_1 + \eta (H \otimes \mathbb{1} - \mathbb{1} \otimes H)} \\ &+ A(u) \otimes B(u) \frac{\delta_2 - \delta_1 + \eta (-\lambda_1 - \mathbb{1} \otimes H)}{\delta_2 - \delta_1 + \eta (H \otimes \mathbb{1} - \mathbb{1} \otimes H)} \end{aligned} \quad (4.25b)$$

$$\begin{aligned} \widetilde{F} \cdot \Delta C(u) \cdot \widetilde{F}^{-1} &= C(u) \otimes A(u) \frac{\delta_2 - \delta_1 + \eta (H \otimes \mathbb{1} - \lambda_2)}{\delta_2 - \delta_1 + \eta (H \otimes \mathbb{1} - \mathbb{1} \otimes H)} \\ &+ A(u) \otimes C(u) \frac{\delta_2 - \delta_1 + \eta (\lambda_1 - \mathbb{1} \otimes H)}{\delta_2 - \delta_1 + \eta (H \otimes \mathbb{1} - \mathbb{1} \otimes H)}. \end{aligned} \quad (4.25c)$$

In particular, it is cocommutative. Cocommutativity of $\Delta D(u)$ follows now using the quantum determinant.

4.2. Towards a universal expression

Finally, we would like to write down the operators Q_{12}^{-1} , respectively, \widetilde{Q}_{12}^{-1} in a form that is independent of the particular representation. Firstly, the recursion formulae (4.4) have to be solved. A case-by-case study of these conditions for small finite-dimensional representations makes it possible to find a solution.

Lemma 4.6. *The coefficients*

$$q_{\ell k} = \prod_{j=0}^{k-1} \frac{\delta_1 - \delta_2 + \eta (\lambda_1 - \lambda_2 - \ell + j)}{\delta_1 - \delta_2 + \eta (-\lambda_1 - \lambda_2 + j)} \quad (4.26)$$

satisfy the recursion relations (4.4).

Recall that $g(x) = \delta_1 - \delta_2 + \eta (\lambda_1 - \lambda_2 + x)$. The proof is a direct computation. For a universal expression, it is not desirable to have a range of the product in (4.26) which depends on the index k . Here k corresponds to the weight of the right factor of the tensor product. The dependence on k can be avoided if one uses quotients of Gamma-functions. In the following we write $H \otimes \mathbb{1}|\ell, k) = (\lambda_1 - \ell)|\ell, k)$ respectively $\mathbb{1} \otimes H|\ell, k) = (\lambda_2 - k)|\ell, k)$.

Proposition 4.7. *The expression*

$$Q_{12}^{-1} = \frac{\Gamma((\delta_1 - \delta_2)/\eta + H \otimes \mathbb{1} - \mathbb{1} \otimes H)}{\Gamma((\delta_1 - \delta_2)/\eta + H \otimes \mathbb{1} - \lambda_2)} \cdot \frac{\Gamma((\delta_1 - \delta_2)/\eta - \lambda_1 - \lambda_2)}{\Gamma((\delta_1 - \delta_2)/\eta - \lambda_1 - \mathbb{1} \otimes H)} \quad (4.27)$$

specializes to the solution (4.26) of the recursion relations on all weight vectors of the representations $V_{\lambda_1}(\delta_1) \otimes V_{\lambda_2}(\delta_2)$.

A completely analogous construction is available for the recursion relations (4.21) for the case where $\Delta A(u)$ is diagonalized. We just summarize the results:

Lemma 4.8. *The coefficients*

$$\tilde{q}_{\ell k} = \prod_{j=1}^{\ell} \frac{\delta_2 - \delta_1 + \eta(\lambda_1 - \lambda_2 + k - \ell + j)}{\delta_2 - \delta_1 + \eta(\lambda_1 + \lambda_2 - \ell + j)} \tag{4.28}$$

satisfy the recursion relations (4.21).

Recall that $\tilde{g}(x) = \delta_2 - \delta_1 + \eta(\lambda_1 - \lambda_2 + x)$. The proof is again a direct computation.

Proposition 4.9. *The expression*

$$\tilde{Q}_{12}^{-1} = \frac{\Gamma((\delta_2 - \delta_1)/\eta + \lambda_1 - \mathbb{1} \otimes H + 1)}{\Gamma((\delta_2 - \delta_1)/\eta + H \otimes \mathbb{1} - \mathbb{1} \otimes H + 1)} \cdot \frac{\Gamma((\delta_2 - \delta_1)/\eta + H \otimes \mathbb{1} + \lambda_2 + 1)}{\Gamma((\delta_2 - \delta_1)/\eta + \lambda_1 + \lambda_2 + 1)} \tag{4.29}$$

specializes to the solution (4.28) of the recursion relations on all weight vectors of representations $V_{\lambda_1}(\delta_1) \otimes V_{\lambda_2}(\delta_2)$.

5. The universal R -matrix of the Yangian $Y(\mathfrak{sl}_2)$

In this section we show how the factorizing twists can be used to calculate R -matrices for the representations $V_{\lambda_1}(\delta_1) \otimes V_{\lambda_2}(\delta_2)$.

5.1. The Gauss decomposition

Khoroshkin and Tolstoy [10] calculate the universal R -matrix of the quantum double $\mathcal{DY}(\mathfrak{sl}_2)$ of the Yangian $Y(\mathfrak{sl}_2)$ and, exploiting the similarities of representations of $\mathcal{DY}(\mathfrak{sl}_2)$ with those of $Y(\mathfrak{sl}_2)$, obtain an expression for the R -matrix that holds on generic evaluation representations of $Y(\mathfrak{sl}_2)$. This R -matrix is presented in its Gauss decomposition

$$R = R_+ R_0 R_- \tag{5.1}$$

(see theorem 5.1 in [10]), where the triangular parts R_+ and R_- simplify according to (6.6) and (6.7) in [10]. In order to compare these results with our notation, we replace $a \mapsto \delta_1/\eta$, $b \mapsto \delta_2/\eta$ and take into account that the Lie algebra \mathfrak{sl}_2 in [10] is written in a Chevalley basis, but in this paper in a Cartan–Weyl basis, i.e. $h \mapsto 2H$. The result from [10] thus reads in our notation

$$R_+ = \sum_{n=0}^{\infty} \frac{\eta^n}{n!} E^n \otimes F^n \left(\prod_{j=1}^n (\delta_1 - \delta_2 + \eta(H \otimes \mathbb{1} - \mathbb{1} \otimes H) + \eta j)^{-1} \right) \tag{5.2}$$

$$R_- = \sum_{n=0}^{\infty} \frac{\eta^n}{n!} \left(\prod_{j=1}^n (\delta_1 - \delta_2 + \eta(H \otimes \mathbb{1} - \mathbb{1} \otimes H) + \eta j)^{-1} \right) F^n \otimes E^n. \tag{5.3}$$

For the specialization of the diagonal part R_0 to evaluation representations, we write (6.13) from [10] in our notation:

$$\begin{aligned} R_0 |\ell, k\rangle &= \frac{\Gamma(\frac{1}{2}[(\delta_1 - \delta_2)/\eta + \lambda_1 - \lambda_2 + k + 1])}{\Gamma(\frac{1}{2}[(\delta_1 - \delta_2)/\eta + \lambda_1 - \lambda_2 + 1])} \cdot \frac{\Gamma(\frac{1}{2}[(\delta_1 - \delta_2)/\eta + \lambda_1 - \lambda_2 + k + 2])}{\Gamma(\frac{1}{2}[(\delta_1 - \delta_2)/\eta - \lambda_1 + \lambda_2 + 1])} \\ &\times \frac{\Gamma(\frac{1}{2}[(\delta_1 - \delta_2)/\eta - \lambda_1 - \lambda_2 + k])}{\Gamma(\frac{1}{2}[(\delta_1 - \delta_2)/\eta + \lambda_1 + \lambda_2 + 2])} \cdot \frac{\Gamma(\frac{1}{2}[(\delta_1 - \delta_2)/\eta - \lambda_1 - \lambda_2 + k + 1])}{\Gamma(\frac{1}{2}[(\delta_1 - \delta_2)/\eta - \lambda_1 - \lambda_2])} \end{aligned}$$

$$\begin{aligned}
 & \times \frac{\Gamma(\frac{1}{2}[(\delta_1 - \delta_2)/\eta + \lambda_1 - \lambda_2 - \ell])}{\Gamma(\frac{1}{2}[(\delta_1 - \delta_2)/\eta + \lambda_1 - \lambda_2 + k - \ell])} \\
 & \cdot \frac{\Gamma(\frac{1}{2}[(\delta_1 - \delta_2)/\eta + \lambda_1 - \lambda_2 - \ell + 1])}{\Gamma(\frac{1}{2}[(\delta_1 - \delta_2)/\eta + \lambda_1 - \lambda_2 + k - \ell + 1])} \\
 & \times \frac{\Gamma(\frac{1}{2}[(\delta_1 - \delta_2)/\eta + \lambda_1 + \lambda_2 - \ell + 1])}{\Gamma(\frac{1}{2}[(\delta_1 - \delta_2)/\eta + \lambda_1 - \lambda_2 + k - \ell + 1])} \\
 & \cdot \frac{\Gamma(\frac{1}{2}[(\delta_1 - \delta_2)/\eta + \lambda_1 + \lambda_2 - \ell + 2])}{\Gamma(\frac{1}{2}[(\delta_1 - \delta_2)/\eta + \lambda_1 - \lambda_2 + k - \ell + 2])} \Big|_{\ell, k}. \tag{5.4}
 \end{aligned}$$

In order to compare this with the results of sections 3 and 4, we need to know how the product $F_{21}^{-1}F_{12}$ of our twists is normalized compared with (5.4). Since our twist acts as the identity on the highest-weight vector, we have to divide R_0 in (5.4) by the character χ of the R -matrix (see (6.14) and the corresponding comments in [10]),

$$\chi = \frac{\Gamma(\frac{1}{2}[(\delta_1 - \delta_2)/\eta + \lambda_1 + \lambda_2 + 1])}{\Gamma(\frac{1}{2}[(\delta_1 - \delta_2)/\eta + \lambda_1 - \lambda_2 + 1])} \cdot \frac{\Gamma(\frac{1}{2}[(\delta_1 - \delta_2)/\eta - \lambda_1 - \lambda_2 + 1])}{\Gamma(\frac{1}{2}[(\delta_1 - \delta_2)/\eta - \lambda_1 + \lambda_2 + 1])}. \tag{5.5}$$

The character χ depends on the representations under study via the highest weights λ_j . It is determined by the nonlinear relations in the definition of the quasi-triangular structure (2.1a) and (2.1b). Since we do not know the form of the coproduct applied to R , we cannot use these conditions, and thus cannot determine χ from our calculation.

The quotient R_0/χ can finally be simplified using

$$\frac{\Gamma((\alpha + m)/2)\Gamma((\alpha + m + 1)/2)}{\Gamma(\alpha/2)\Gamma((\alpha + 1)/2)} \cdot \frac{\Gamma(\beta/2)\Gamma((\beta + 1)/2)}{\Gamma((\beta + m)/2)\Gamma((\beta + m + 1)/2)} = \frac{\Gamma(\alpha + m)}{\Gamma(\alpha)} \cdot \frac{\Gamma(\beta)}{\Gamma(\beta + m)} \tag{5.6}$$

where $m \in \mathbb{N}_0$, yielding

$$\begin{aligned}
 R_0/\chi &= \frac{\Gamma((\delta_1 - \delta_2)/\eta + \lambda_1 - \mathbb{1} \otimes H + 1)}{\Gamma((\delta_1 - \delta_2)/\eta + H \otimes \mathbb{1} - \mathbb{1} \otimes H + 1)} \cdot \frac{\Gamma((\delta_1 - \delta_2)/\eta + H \otimes \mathbb{1} + \lambda_2 + 1)}{\Gamma((\delta_1 - \delta_2)/\eta + \lambda_1 + \lambda_2 + 1)} \\
 & \times \frac{\Gamma((\delta_1 - \delta_2)/\eta + H \otimes \mathbb{1} - \lambda_2)}{\Gamma((\delta_1 - \delta_2)/\eta + H \otimes \mathbb{1} - \mathbb{1} \otimes H)} \cdot \frac{\Gamma((\delta_1 - \delta_2)/\eta - \lambda_1 - \mathbb{1} \otimes H)}{\Gamma((\delta_1 - \delta_2)/\eta - \lambda_1 - \lambda_2)}. \tag{5.7}
 \end{aligned}$$

5.2. The factorization

It is apparent that the diagonal part (5.7) is factorized by the diagonal part of the twist:

$$R_0/\chi = Q_{21}^{-1}Q_{12}. \tag{5.8}$$

The expression for Q_{12} can be read off from (4.27). The coefficients of Q_{21}^{-1} are obtained from (4.26) by exchanging $k \leftrightarrow \ell$, $\delta_1 \leftrightarrow \delta_2$ and $\lambda_1 \leftrightarrow \lambda_2$:

$$\prod_{j=0}^{\ell-1} \frac{\delta_2 - \delta_1 + \eta(\lambda_2 - \lambda_1 - k + j)}{\delta_2 - \delta_1 + \eta(-\lambda_2 - \lambda_1 + j)} = \prod_{j=0}^{\ell-1} \frac{\delta_1 - \delta_2 + \eta(\lambda_1 - \lambda_2 + k - \ell + 1 + j)}{\delta_1 - \delta_2 + \eta(\lambda_1 + \lambda_2 - \ell + 1 + j)} \tag{5.9}$$

from which Q_{21}^{-1} can be calculated

$$Q_{21}^{-1} = \frac{\Gamma((\delta_1 - \delta_2)/\eta + \lambda_1 - \mathbb{1} \otimes H + 1)}{\Gamma((\delta_1 - \delta_2)/\eta + H \otimes \mathbb{1} - \mathbb{1} \otimes H + 1)} \cdot \frac{\Gamma((\delta_1 - \delta_2)/\eta + H \otimes \mathbb{1} + \lambda_2 + 1)}{\Gamma((\delta_1 - \delta_2)/\eta + \lambda_1 + \lambda_2 + 1)} \tag{5.10}$$

confirming (5.8).

Finally, from (3.24) we find $F_{21}^{-1} = R_+ Q_{21}^{-1}$ and from (3.26) that $F_{12} = Q_{12} R_-$. This completes the factorization

$$F_{21}^{-1} F_{12} = R_+ Q_{21}^{-1} Q_{12} R_- = R = R_+ (R_0/\chi) R_- \tag{5.11}$$

Likewise for the twist \widehat{F} which was obtained diagonalizing $\Delta A(u)$, we have

$$\widehat{F}_{12}^{-1} = F_{21}^{-1} \Big|_{\delta_1 \leftrightarrow \delta_2} \tag{5.12}$$

from which an analogous result follows.

The factorization of the diagonal part of the R -matrix in (5.8) is remarkably simple. In particular, it does not involve any of the complications which have initially been conjectured [7]. There it was speculated that the universal forms of the two factors F_{12}^{-1} and F_{21} might not, in general, be related by the usual inversion and swapping of tensor factors, but only related in finite-dimensional representations. Actually, the factorization is simple for all highest-weight representations.

5.3. A discrete symmetry

There exists an alternative choice for the diagonal part of the twist in addition to that constructed in section 4. In proposition 4.1, it is possible to use coefficients $\widehat{q}_{\ell k}$ instead of the $q_{\ell k}$ which satisfy alternative recursion relations

$$\frac{\widehat{q}_{\ell+1,k}}{\widehat{q}_{\ell k}} = \frac{g(2\lambda_2 - \ell)}{g(k - \ell)} \quad \frac{\widehat{q}_{\ell k}}{\widehat{q}_{\ell,k-1}} = \frac{g(k - \ell)}{g(k)} \tag{5.13}$$

This leads to another twist \widehat{F} having the same triangular, but a different diagonal part. The new coproduct differs from that in corollary 4.2 by the ordering of operators:

$$\begin{aligned} \widehat{F} \cdot \Delta B(u) \cdot \widehat{F}^{-1} &= \frac{\delta_1 - \delta_2 + \eta(H \otimes \mathbb{1} - \lambda_2)}{\delta_1 - \delta_2 + \eta(H \otimes \mathbb{1} - \mathbb{1} \otimes H)} B(u) \otimes D(u) \\ &+ \frac{\delta_1 - \delta_2 + \eta(\lambda_1 - \mathbb{1} \otimes H)}{\delta_1 - \delta_2 + \eta(H \otimes \mathbb{1} - \mathbb{1} \otimes H)} D(u) \otimes B(u) \end{aligned} \tag{5.14a}$$

$$\begin{aligned} \widehat{F} \cdot \Delta C(u) \cdot \widehat{F}^{-1} &= \frac{\delta_1 - \delta_2 + \eta(H \otimes \mathbb{1} + \lambda_2)}{\delta_1 - \delta_2 + \eta(H \otimes \mathbb{1} - \mathbb{1} \otimes H)} C(u) \otimes D(u) \\ &+ \frac{\delta_1 - \delta_2 + \eta(-\lambda_1 - \mathbb{1} \otimes H)}{\delta_1 - \delta_2 + \eta(H \otimes \mathbb{1} - \mathbb{1} \otimes H)} D(u) \otimes C(u). \end{aligned} \tag{5.14b}$$

The recursion relations (5.13) are solved by

$$\widehat{q}_{\ell k} = \prod_{j=0}^{\ell-1} \frac{\delta_1 - \delta_2 + \eta(\lambda_1 + \lambda_2 - \ell + 1 + j)}{\delta_1 - \delta_2 + \eta(\lambda_1 - \lambda_2 + k - \ell + 1 + j)} \tag{5.15}$$

A comparison with (5.9) and (5.10) shows that the corresponding diagonal operator

$$\widehat{Q}_{12}^{-1} | \ell, k \rangle = \widehat{q}_{\ell k} | \ell, k \rangle \tag{5.16}$$

is just $\widehat{Q}_{12}^{-1} = Q_{21}$.

This shows that we are free to associate the factors Q_{21}^{-1} and Q_{12} of R_0/χ in (5.11) with either one of the triangular factors to form a factorizing twist. For \widehat{F} we have $\widehat{F}_{21}^{-1} = R_+ Q_{12}$ and $\widehat{F}_{12} = Q_{21}^{-1} R_-$ such that the factorization reads

$$\widehat{F}_{21}^{-1} \widehat{F}_{12} = R_+ Q_{12} Q_{21}^{-1} R_- = R_+ (R_0/\chi) R_- \tag{5.17}$$

The coproduct found in [7] using the modified functional Bethe ansatz is the one given in (5.14). The twist found in that paper is our \widehat{F} . We give, in addition, the explicit form of its diagonal part $\widehat{Q}_{12}^{-1} = Q_{21}$ as the inverse of (5.10).

5.4. Existence of the factorizing twist in particular representations

Finally, we comment on the question for which representations $V_{\lambda_1}(\delta_1) \otimes V_{\lambda_2}(\delta_2)$ the twist F_{12} and its inverse exist. We restrict ourselves to the finite-dimensional case where corresponding results for the existence of the R -matrices are known. We recall that the twist is to factorize the R -matrix in the relation

$$\Delta^{\text{op}} X(u) = R(\delta_1 - \delta_2) \Delta X(u) R^{-1}(\delta_1 - \delta_2).$$

Remark 5.1.

- (a) The R -matrix $R(\delta_1 - \delta_2)$ as a function of $\delta_1 - \delta_2$ is well defined if and only if the representation $V_{\lambda_1}(\delta_1) \otimes V_{\lambda_2}(\delta_2)$ is irreducible (see theorem 2.7).
- (b) It is thus reasonable to expect that the twist exists at most in the irreducible representations, but there might be additional poles in the twist which cancel only in the product $F_{21}^{-1} F_{12} = R_{12}$.

In the following we analyse the structure of poles in F_{12} and F_{12}^{-1} . We find that certain factors from the denominator of the triangular part cancel with the numerator of the diagonal part.

Lemma 5.2. The coefficients of Q_{12}^{-1} , see (4.26), can be written as

$$q_{\ell k} = \frac{\prod_{j=\max\{2\lambda_1-\ell+1, k+1\}}^{k+2\lambda_1-\ell} (\delta_1 - \delta_2 + \eta(-\lambda_1 - \lambda_2 + j - 1))}{\prod_{j=1}^{\min\{k, 2\lambda_1-\ell\}} (\delta_1 - \delta_2 + \eta(-\lambda_1 - \lambda_2 + j - 1))} \tag{5.18}$$

$$= \prod_{j=1}^{\min\{k, 2\lambda_1-\ell\}} \frac{\delta_1 - \delta_2 + \eta(\lambda_1 - \lambda_2 + k - \ell - j)}{\delta_1 - \delta_2 + \eta(-\lambda_1 - \lambda_2 + j - 1)} \tag{5.19}$$

where the numerator and denominator have no common factor.

Proposition 5.3. The expression F_{12}^{-1} , see (3.24), is well defined for all finite-dimensional irreducible representations $V_{\lambda_1}(\delta_1) \otimes V_{\lambda_2}(\delta_2)$.

Proof. First, we show that all factors in the denominator of the triangular part of F_{12}^{-1} cancel with the numerator of Q_{12}^{-1} . Poles can be present only if $F^n \otimes E^n|\ell, k \neq 0$, but this implies $n \leq \min\{k, 2\lambda_1 - \ell\}$. The denominator of the triangular part, see (3.24), applied to $|\ell, k$ is

$$\prod_{j=1}^n (\delta_1 - \delta_2 + \eta(\lambda_1 - \lambda_2 + k - \ell - j)). \tag{5.20}$$

Since $n \leq \min\{k, 2\lambda_1 - \ell\}$, it cancels with the numerator of Q_{12}^{-1} as given in (5.19).

The denominator of Q_{12}^{-1} has a pole if and only if

$$\frac{\delta_1 - \delta_2}{\eta} = \lambda_1 + \lambda_2 - j + 1 \tag{5.21}$$

where $1 \leq j \leq \min\{k, 2\lambda_1 - \ell\}$. However, since $k \leq 2\lambda_2$ and $\ell \geq 0$, this implies $1 \leq j \leq \min\{2\lambda_1, 2\lambda_2\}$ for which $V_{\lambda_1}(\delta_1) \otimes V_{\lambda_2}(\delta_2)$ is reducible (theorem 2.7). \square

Similarly, we can study F_{12} , see (3.26). In this case not all factors in the denominator of the triangular part are cancelled.

Lemma 5.4. The expression F_{12} in (3.26) has poles at

$$\frac{\delta_1 - \delta_2}{\eta} = \lambda_1 + \lambda_2 - j + 1 \tag{5.22}$$

for precisely the values $j \in \{2, 3, \dots, 2\lambda_1 + 2\lambda_2\}$.

Proof. The denominator of the triangular part of F_{12} acting on $|\ell, k\rangle$ is given by

$$\prod_{j=1}^n (\delta_1 - \delta_2 + \eta (\lambda_1 - \lambda_2 + k - \ell - 2n + j))$$

$$= \prod_{j=2+(2\lambda_1-\ell)+k-2n}^{1+(2\lambda_1-\ell)+k-n} (\delta_1 - \delta_2 + \eta (-\lambda_1 - \lambda_2 + j - 1)) \tag{5.23}$$

where $1 \leq n \leq \min\{k, 2\lambda_1 - \ell\}$. Some factors cancel with the numerator of Q_{12} (see the expression for Q_{12}^{-1} in (5.18)). The remaining denominator of the triangular part is

$$\prod_{j=\max\{2+(2\lambda_1-\ell)+k-2n, \min\{k+1, 2\lambda_1-\ell+1\}\}}^{1+(2\lambda_1-\ell)+k-n} (\delta_1 - \delta_2 + \eta (-\lambda_1 - \lambda_2 + j - 1)) \tag{5.24}$$

since $2 + (2\lambda_1 - \ell) + k - 2n > 1$ and $1 + (2\lambda_1 - \ell) + k - n \geq \max\{k + 1, (2\lambda_1 - \ell) + 1\} > \min\{k, 2\lambda_1 - \ell\}$.

The denominator of the diagonal part Q_{12} is

$$\prod_{j=\max\{2\lambda_1-\ell+1, k+1\}}^{k+2\lambda_1-\ell} (\delta_1 - \delta_2 + \eta (-\lambda_1 - \lambda_2 + j - 1)). \tag{5.25}$$

Poles in F_{12} can arise from both (5.24) and (5.25). If $k = 0$ or $\ell = 2\lambda_1$, then $n = 0$, and both products (5.24) and (5.25) are empty.

If $k \geq 1$ and $\ell \leq 2\lambda_1 - 1$, there is an $n = 1$ contribution to the product (5.24) which extends in this case from $j = 2$ to $j = (2\lambda_1 - \ell) + k$. The second product (5.25) runs from $j = \max\{2\lambda_1 - \ell + 1, k + 1\} \geq 2$ to $j = (2\lambda_1 - \ell) + k$. For arbitrary $n \leq \min\{k, 2\lambda_1 - \ell\}$, these bounds are not exceeded. \square

Corollary 5.5. *The expression F_{12} in (3.26) is well defined on $V_{\lambda_1}(\delta_1) \otimes V_{\lambda_2}(\delta_2)$ if and only if*

$$\frac{\delta_1 - \delta_2}{\eta} \notin \{-\lambda_1 - \lambda_2 + 1, -\lambda_1 - \lambda_2 + 2, \dots, \lambda_1 + \lambda_2 - 1\}. \tag{5.26}$$

In particular, there exist irreducible representations $V_{\lambda_1}(\delta_1) \otimes V_{\lambda_2}(\delta_2)$ for which F_{12} does not exist.

Remark 5.6. *According to theorem 2.7, the finite-dimensional representation $V_{\lambda_1}(\delta_1) \otimes V_{\lambda_2}(\delta_2)$ is irreducible if and only if*

$$\frac{\delta_1 - \delta_2}{\eta} \notin \{-\lambda_1 - \lambda_2, \dots, -|\lambda_1 - \lambda_2| - 1; |\lambda_1 - \lambda_2| + 1, \dots, \lambda_1 + \lambda_2\} \tag{5.27}$$

where the set does not include values around zero $-|\lambda_1 - \lambda_2|, \dots, |\lambda_1 + \lambda_2|$. For these values there exists F_{12}^{-1} , but not F_{12} .

The simplest example is the representation $V_\lambda(\delta) \otimes V_\lambda(\delta)$, where both $R(0) = P$ and $R^{-1}(0) = P$ are given by the operator which exchanges the tensor factors, $P(a \otimes b) = b \otimes a$. Obviously,

$$\Delta^{\text{op}} X(u) = P \Delta X(u) P. \tag{5.28}$$

However, P is ‘too symmetric’ to be factorized in an expression like $P = F_{21}^{-1} F_{12}$.

5.5. On tensor products of evaluation representations

Having obtained the R -matrix on generic evaluation representations $V_{\lambda_1}(\delta_1) \otimes V_{\lambda_2}(\delta_2)$, the R -matrices on tensor products with more than two factors are determined by the axioms

$$(\Delta \otimes \text{id})(R) = R_{13}R_{23}$$

$$(\text{id} \otimes \Delta)(R) = R_{13}R_{12}$$

of the quasi-triangular structure. It is thus possible to calculate the R -matrices for tensor products of evaluation representations even without knowing the action of the coproduct.

The R -matrices obtained this way, however, agree only up to the character χ with the R -matrix which is obtained by representing the universal R -matrix directly on a tensor product of evaluation representations. The analysis in [10] shows, however, that the characters χ are well understood so that this is not a serious drawback.

Our method thus determines R -matrices for all tensor products of evaluation representations. These include, in particular, all finite-dimensional irreducible representations of $Y(\mathfrak{sl}_2)$ and those representations of interest in applications to integrable systems.

6. Conclusion and outlook

In this paper we have presented an elementary direct calculation of the factorizing twist of $Y(\mathfrak{sl}_2)$ which is universal for all evaluation representations. Having calculated the twist, it is possible to recover the universal R -matrix specialized to a generic evaluation representation. It appears automatically in a canonical form, being Gauss decomposed as an upper-triangular times a diagonal times a lower-triangular part.

The fact that this approach is successful underlines the importance of studying the quantum groups such as the Yangian $Y(\mathfrak{g})$ or the quantized envelopes of the affine Lie algebras $U_q(\widehat{\mathfrak{g}})$ in view of their pseudo-triangularity. The factorizing twist seems more fundamental and even more accessible than the universal R -matrix. This is very relevant for the study of analogous constructions for Lie algebras \mathfrak{g} of higher rank as well as for their applications to quantum integrable systems. A thorough understanding of factorizing twists for \mathfrak{g} of higher rank can be expected to simplify the nested version of the algebraic Bethe ansatz dramatically as indicated in [6].

At present the form of the factorizing twist on the abstract Yangian algebra is not known. It may well be more difficult to deal with than is the universal R -matrix because the twist exists for fewer representations than R -matrices do. We thus expect only a pseudo-twist.

However, in any case the factorizing twists provide the Yangian $Y(\mathfrak{sl}_2)$ with an additional, very restrictive structure which has not been fully exploited in the analysis of the algebra yet.

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